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CONVERGENCE-SUBTRAHENDS FOR THE TRIGONOMETRICAL FUNCTIONS EXPRESSED IN INFINITE SUMS.

By PROF. R. D. BOHANNAN, University of Virginia.

If $F(z)$ is a function of a complex variable, $z = x + iy$, in general finite and continuous, but infinite on the points, $z = a_1, a_2, a_3, \ldots a_n$, and so infinite that $\lim (z - a_n) F(z) = A_n$ when $z = a_n$, where A_n is not infinite, then

$$F(z) = \sum_1^n \frac{A_n}{z - a_n} + G(z),$$

where $G(z)$ is a function not infinite for any finite value of z. When the number of points on which $F(z)$ is infinite after the manner stated, is infinite, then

$$F(z) = \lim \sum_1^\infty \left\{ \frac{A_n}{z - a_n} + S(z) \right\} + G'(z).*$$

For any particular function, that which offers the greatest difficulty is the determination of the functions $G(z)$ and $G'(z)$. For this there seems to be no general method, except in so far as the contour-integration method of Cauchy is applicable; and in the application of this, great care is necessary.†

In all cases where an infinite sum is present, $S(z)$, called, conveniently, the Convergence-Subtrahend, must be so determined as to render the infinite sum convergent. Generally a number of functions may be found which will do this, thus making $G'(z)$ also variable. For a new function, after $S(z)$ has been determined, $G'(z)$ is quite arbitrary; for an old function whose properties are well known, $G'(z)$ must be so determined, in connection with the infinite sum, as to bring out these properties.

*See Weierstrass's *Zur Theorie der eindeutigen analytischen Functionen*, Berlin, 1876; the writings of Mittag-Leffler based on this pamphlet; and Schering's *Das Anschliessen einer Function an algebraische Functionen in unendlich vielen Stellen*, Goettingen, 1880.

†Compare Hermite's *Cours 1881–82*, XIIe Leçon, and his note on same in *American Journal of Mathematics* for September, 1883.

The object of this paper is to give a convenient form to $S(z)$ for the functions

$$\frac{1}{\sin z}, \frac{1}{\cos z}, \tan z, \cot z.$$

The only *essentially* singular point which these functions have, is "the point at infinity." Consequently here the infinite sums must be made *absolutely* convergent for all finite values of absolute z, other than those for which the functions become infinite. It will be assumed in what follows, that, for each of these functions, $G'(z)$, in connection with the $S(z)$ which we shall find, is zero. This can be proven, either by contour-integration,* or by showing, on the assumption that $G'(z)=0$, that the absolutely convergent infinite sums have all the properties of the functions they are made to represent, and agree with them in value, on points selected at random.

$$\frac{1}{\sin z}.$$

$\frac{1}{\sin z} = \infty$ when $z = m\pi$, m being 0 or any positive or negative integer, and $\lim \frac{z - m\pi}{\sin z} = (-1)^m$ when $z = m\pi$; so that

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_1^\infty{}' \left[\frac{(-1)^m}{z - m\pi} + S_1(z) \right] + \sum_1^\infty{}' \left[\frac{(-1)^m}{z + m\pi} + S_2(z) \right].$$

The sums $\sum_1^\infty{}' \frac{(-1)^m}{z - m\pi}$ and $\sum_1^\infty{}' \frac{(-1)^m}{z + m\pi}$ are not absolutely convergent. The infinite sums in the value of $\frac{1}{\sin z}$ are most conveniently rendered absolutely convergent if we determine $S_1(z)$ and $S_2(z)$ so that

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_1^\infty{}' (-1)^m \left[\frac{1}{z - m\pi} + \frac{1}{m\pi} \right] + \sum_1^\infty{}' (-1)^m \left[\frac{1}{z + m\pi} - \frac{1}{m\pi} \right].$$

This is identical in value with the usual expression

$$\frac{1}{\sin z} = \frac{1}{z} + 2z \sum_1^\infty{}' \frac{(-1)^m}{z^2 - m^2\pi^2},$$

but the former shows, almost on its face, the periodicity of $\sin z$; and the periodicity of the trigonometrical functions is their most interesting property.

*See Briot et Bouquet's *Fonctions Doublement Periodiques*, p. 123; also their *Fonctions Elliptiques*, p. 281; also Hermite's *Cours*, p. 79.

$$\frac{1}{\cos z}.$$

$\frac{1}{\cos z} = \infty$ when $z = \pm (2m+1)\frac{\pi}{2}$, and $\lim \frac{(2m+1)\frac{\pi}{2} - z}{\cos z} = (-1)^m$.

Hence, corresponding to the formula for $\frac{1}{\sin z}$, we have

$$\frac{1}{\cos z} = \frac{1}{\frac{\pi}{2} - z} + \sum_1^\infty (-1)^m \left(\frac{1}{\frac{\pi}{2} - z - m\pi} + \frac{1}{m\pi} \right) + \sum_1^\infty (-1)^m \left(\frac{1}{\frac{\pi}{2} - z + m\pi} - \frac{1}{m\pi} \right).$$

This formula shows the periodicity of $\cos z$ and the connection between $\sin z$ and $\cos z$.

Briot and Bouquet, both in their *Fonctions Doublement Périodiques*, p. 125, and in their *Fonctions Elliptiques*, p. 285, obtain, by grouping the terms of an infinite sum, the expression

$$\frac{1}{\cos z} = \pi \sum_0^\infty \frac{(2m+1)(-1)^m}{(2m+1)^2 \frac{\pi^2}{4} - z^2}.$$

This is not an allowable formula, for it is not absolutely convergent for any value of z, as may be readily shown.

Similar formulæ for $\tan z$ and $\cot z$ are

$$\tan z = \frac{1}{\frac{\pi}{2} - z} - \frac{1}{\frac{\pi}{2} + z} + \sum_1^\infty \left(\frac{1}{\frac{\pi}{2} - z + m\pi} - \frac{1}{m\pi} \right) + \sum_1^\infty \left(\frac{1}{\frac{\pi}{2} - z - m\pi} + \frac{1}{m\pi} \right)$$

and

$$\cot z = \frac{1}{z} + \sum_1^\infty \left(\frac{1}{z - m\pi} + \frac{1}{m\pi} \right) + \sum_1^\infty \left(\frac{1}{z + m\pi} - \frac{1}{m\pi} \right).$$

ON THE LUNAR INEQUALITIES PRODUCED BY THE MOTION OF THE ECLIPTIC.

By Dr. G. W. Hill, Washington, D. C.

[CONTINUED FROM PAGE 31.]

II.

The direct action of the planets produces in the motion of the moon terms which have nearly the same periods as those we have been considering. To complete the subject it is necessary to derive these and add them to those just obtained. If $\delta' R$ denote the part of R which is due to the action of the planet m'', and Δ the distance of the latter from the earth, two accents being used to denote quantities which belong to the planet,

$$\delta' R = m'' \left\{ [\Delta^2 - 2(x''x + y''y + z''z) + r^2]^{-\frac{1}{2}} - \frac{x''x + y''y + z''z}{r''^3} \right\}.$$

Or, with sufficient approximation,

$$\delta' R = \frac{m''}{2} \left\{ 3\frac{(x''x + y''y + z''z)^2}{\Delta^5} - \frac{r^2}{\Delta^3} \right\}.$$

The only part of $\delta' R$ which can produce terms we are in search of is that which has z'' for a factor: thus we may take

$$\delta' R = 3m'' \frac{(x''x + y''y)\, z''z}{\Delta^5}.$$

But, with sufficient approximation,

$$\begin{aligned}
x'' &= a'' \cos(\varepsilon'' + n''t) + a' \cos(\varepsilon' + n't), \\
y'' &= a'' \sin(\varepsilon'' + n''t) + a' \sin(\varepsilon' + n't), \\
z'' &= a''\gamma'' \sin(\varepsilon'' + n''t - \Omega''), \\
\Delta^{-5} &= \frac{1}{2} A_0 + A_1 \cos[\varepsilon'' - \varepsilon' + (n'' - n')t] + \ldots, \\
x''x + y''y &= a''r \cos(\lambda - \varepsilon'' - n''t) + a'r \cos(\lambda - \varepsilon' - n't).
\end{aligned}$$

Preserving only terms which are needed,

$$\begin{aligned}
(x''x + y''y)\, z'' &= \frac{1}{2} a''\gamma'' r \left\{ a'' \sin(\lambda - \Omega'') \right. \\
&\qquad \left. + a' \sin[\lambda - \Omega'' + \varepsilon'' - \varepsilon' + (n'' - n')t] \right\}, \\
\frac{(x''x + y''y)\, z''z}{\Delta^5} &= \frac{1}{4} a''\gamma''(a''A_0 + a'A_1)\, rz \sin(\lambda - \Omega'').
\end{aligned}$$

Consequently

$$\begin{aligned}\delta' R &= \frac{3}{4} m'' a'' \gamma'' (a'' A_0 + a' A_1) rz \sin(\lambda - \Omega'') \\ &= \frac{3}{4}\frac{m''}{m'} m^2 a'^3 a'' \gamma'' (a'' A_0 + a' A_1) rz \sin(\lambda - \Omega'') \\ &= -2Krz \sin(\lambda - \Omega'').\end{aligned}$$

For an inferior planet

$$A_0 = a'^{-5} b_{\frac{3}{2}}^{(0)}, \qquad A_1 = -a'^{-5} b_{\frac{3}{2}}^{(1)},$$

and for a superior one

$$A_0 = a''^{-5} b_{\frac{3}{2}}^{(0)}, \qquad A_1 = -a''^{-5} b_{\frac{3}{2}}^{(1)}.$$

But

$$b_{\frac{3}{2}}^{(0)} = \frac{(1+\alpha^2) b_{\frac{3}{2}}^{(0)} + \frac{2}{3} \alpha b_{\frac{3}{2}}^{(1)}}{(1-\alpha^2)^2}, \quad b_{\frac{3}{2}}^{(1)} = \frac{2\alpha b_{\frac{3}{2}}^{(0)} + \frac{1}{3}(1+\alpha^2) b_{\frac{3}{2}}^{(1)}}{(1-\alpha^2)^2}.$$

Consequently, for an inferior planet,

$$K = \frac{3}{8}\frac{m''}{m'} m^2 \gamma'' \frac{\alpha}{1-\alpha^2}\left(\alpha b_{\frac{3}{2}}^{(0)} + \frac{1}{3} b_{\frac{3}{2}}^{(1)}\right),$$

and, for a superior,

$$K = -\frac{3}{8}\frac{m''}{m'} m^2 \gamma'' \frac{\alpha^3}{1-\alpha^2}\left(b_{\frac{3}{2}}^{(0)} + \frac{1}{3}\alpha b_{\frac{3}{2}}^{(1)}\right).$$

To determine δz we shall have the equation

$$\frac{d^2 \delta z}{d\zeta^2} + A \delta z = -2Kr \sin(\lambda - \Omega'').$$

Making $\varepsilon + nt - \Omega'' = \zeta'$,

$$Kr \sin(\lambda - \Omega'') = \left(1 - \frac{m^2}{6}\right) K \sin \zeta' - \frac{19}{16} m^2 K \sin(\zeta' - 2\tau).$$

Since K is much smaller than $\frac{\pi}{n'}$, we shall content ourselves with one order of approximation less in the factors which multiply it than in those which multiply $\frac{\pi}{n'}$. With this restriction, it will be readily seen that the value of δz is obtained simply by writing K and ζ' respectively for $\frac{\pi}{n}$ and ζ in the formula previously obtained. Thus

$$\delta z = -\left(\frac{4}{3} m^{-2} + \frac{1}{2} m^{-1} + \frac{31}{9}\right) K \sin \zeta'$$

$$+\left(\frac{1}{2}m^{-1}+\frac{25}{12}\right)K\sin(\zeta'-2\tau)$$
$$-\frac{1}{4}K\sin(\zeta'+2\tau).$$

As regards the differential equations which determine $r\partial r$ and $\partial\lambda$, it is evident that they remain the same as before, with the exceptions that K and ζ' everywhere take the place of $\frac{\pi}{n}$ and ζ; and in U', in place of $2\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}$, must be put $4\partial' R=0$, and that consequently U' in this case becomes

$$U'=\gamma K[2\cos(\zeta'-\eta-2\tau)+2\cos(\zeta'-\eta+2\tau)-\cos(\zeta'+\eta-2\tau)];$$

and in U'', in place of $2\frac{\pi}{n}\int xdz$, must be put

$$-2K\int rz\cos(\lambda-\Omega'')\,d\zeta=-\frac{3}{16}\gamma K\cos(\zeta'+\eta-2\tau),$$

whence U'' in this case becomes

$$U''=\gamma K\left\{\left(\frac{1}{2}m^{-1}+\frac{1}{2}\right)\cos(\zeta'-\eta-2\tau)+\left(\frac{1}{2}m^{-1}+\frac{5}{16}\right)\cos(\zeta'-\eta+2\tau)\right.$$
$$\left.+\left(\frac{4}{3}m^{-2}+\frac{1}{2}m^{-1}\right)\cos(\zeta'+\eta)-\left\{\frac{1}{2}m^{-1}+\frac{7}{3}\right\}\cos(\zeta'+\eta-2\tau)\right\};$$

and, in the differential equation for determining the coefficient of $\sin(\zeta-\eta)$ in $\partial\lambda$, in place of $-4\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}$, must be put

$$-7\partial' R=7\gamma K\cos(\zeta'-\eta).$$

Making use of similar expressions for $r\partial r$ and $\partial\lambda$ as were used in the former case, we obtain the equations of condition

$$-3B_2+3m^2C_1=2,$$
$$-3B_3-3m^2C_1=2,$$
$$-B_5-\frac{3}{2}m^2C_4=1,$$
$$\left.\begin{aligned}&\left(\frac{3}{4}m^2-\frac{9}{32}m^3\right)C_1-\frac{9}{2}m^2(C_2-C_3)\\&-\frac{57}{2}m^3C_1+6m(B_2-B_3+C_2+C_3)\end{aligned}\right\}=-\frac{1}{2}+\frac{9}{16}m,$$
$$(2-2m)C_2-\frac{3}{4}m^2C_1-2B_2=-\frac{1}{2}m^{-1}-\frac{1}{2},$$
$$(2-2m)C_3-\frac{3}{4}m^2C_1+2B_3=\frac{1}{2}m^{-1}+\frac{5}{16},$$

$$2C_4 = \frac{4}{3}m^{-2} + \frac{1}{2}m^{-1},$$

$$2mC_5 - \left(\frac{3}{4}m + \frac{55}{32}m^2\right) C_4 + 2B_5 = -\frac{1}{2}m^{-1} - \frac{7}{3}.$$

These equations are the same as we obtained in the case of the inequalities produced by the motion of the ecliptic, with the single exception of that which determines C_1; and, being solved, they give

$$\begin{aligned}\delta\lambda = &-\left(\frac{2}{3}m^{-2} + \frac{3}{2}m^{-1}\right)\gamma K \sin(\zeta' - \eta)\\ &-\frac{1}{4}m^{-1}\gamma K \sin(\zeta' - \eta - 2\tau)\\ &+\frac{1}{4}m^{-1}\gamma K \sin(\zeta' - \eta + 2\tau)\\ &+\left(\frac{2}{3}m^{-2} + \frac{1}{4}m^{-1}\right)\gamma K \sin(\zeta' + \eta)\\ &+\frac{3}{2}m^{-1}\gamma K \sin(\zeta' + \eta - 2\tau).\end{aligned}$$

The expression for the inequalities in latitude is

$$\begin{aligned}\delta_1\beta = &-\left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{11}{3}\right) K \sin\zeta'\\ &+\left(\frac{1}{2}m^{-1} + \frac{17}{12}\right) K \sin(\zeta' - 2\tau)\\ &-\frac{11}{12} K \sin(\zeta' + 2\tau).\end{aligned}$$

III.

It remains only to transform the foregoing formulæ into numerical results According to Hansen and Olufsen (*Tables du Soleil, Introduction*),

$$\pi \sin \Pi = +\,0''.053916, \qquad \pi \cos \Pi = -\,0''.467839,$$

whence

$$\pi = 0''.470903, \text{ and } \Pi = 173^\circ 25' 34''.$$

Also

$$\begin{aligned}n &= 17325225'',\\ m &= 0.074801,\\ \gamma &= 0.089673,\\ e &= 0.054731.\end{aligned}$$

Substituting these values, the inequalities produced by the motion of the ecliptic are

$$\begin{aligned}
\delta\lambda = &+ 0''.2952 \sin(\zeta - \eta) + 0''.0000 \sin(\zeta - \eta - 2\tau) \\
&+ 0''.0045 \sin(\zeta - \eta + 2\tau) + 0''.0616 \sin(\zeta + \eta) \\
&+ 0''.0089 \sin(\zeta + \eta - 2\tau) + 0''.0004 \sin(\zeta + \eta + 2\tau) \\
&+ 0''.0000 \sin(\zeta + \eta - 4\tau), \\
\delta\beta = &- 1''.4001 \sin\zeta + 0''.0469 \sin(\zeta - 2\tau) \\
&- 0''.0064 \sin(\zeta + 2\tau) + 0''.0001 \sin(\zeta - 4\tau) \\
&+ 0''.0757 \sin(\zeta - \xi) - 0''.0768 \sin(\zeta + \xi) \\
&+ 0''.0088 \sin(\zeta - 2\tau + \xi) - 0''.0137 \sin(\zeta + 2\tau - \xi) \\
&+ 0''.0022 \sin(\zeta - 2\tau - \xi) - 0''.0007 \sin(\zeta + 2\tau + \xi) \\
&+ 0''.0003 \sin(\zeta - 4\tau + \xi).
\end{aligned}$$

To compute the terms due to the direct action of the planets we take for Venus,

$$\frac{m''}{m'} = \frac{1}{408134}, \qquad \gamma'' = \tan(3^\circ\ 23'\ 34''), \qquad \Omega'' = 75^\circ\ 21',$$

for Mars,

$$\frac{m''}{m'} = \frac{1}{3200900}, \qquad \gamma'' = \tan(1^\circ\ 51'), \qquad \Omega'' = 48^\circ\ 24',$$

for Jupiter,

$$\frac{m''}{m} = \frac{1}{1050}, \qquad \gamma'' = \tan(1^\circ\ 18'\ 35''), \qquad \Omega'' = 98^\circ\ 57',$$

for Saturn,

$$\frac{m''}{m} = \frac{1}{3512}, \qquad \gamma'' = \tan(2^\circ\ 29'), \qquad \Omega'' = 112^\circ\ 21'.$$

The quantities depending on the ratio of the mean distances are taken from Runkle's *Tables of the Coefficients of the Perturbative Function.* Thus we obtain for the several planets, in their order, the values of $\log K$ expressed in seconds of arc: —

$$\begin{aligned}
\log K &= 96.9867, \\
\log K &= 95.2450_n, \\
\log K &= 96.1878_n, \\
\log K &= 95.1081_n.
\end{aligned}$$

Then the action of Venus produces the following terms: —

$$\begin{aligned}
\delta\lambda &= - 0''.0121 \sin(\zeta' - \eta) + 0''.0106 \sin(\zeta' + \eta), \\
\delta\beta &= - 0''.2412 \sin\zeta' \qquad\qquad + 0''.0078 \sin(\zeta' - 2\tau).
\end{aligned}$$

The action of Mars produces the terms

$$\delta\lambda = +\,0''.0003 \sin(\zeta'' - \eta) - 0''.0002 \sin(\zeta' + \eta),$$
$$\delta_1\beta = +\,0''.0044 \sin \zeta''$$

The action of Jupiter produces the terms

$$\delta\lambda = +\,0''.0019 \sin(\zeta''' - \eta) - 0''.0016 \sin(\zeta''' + \eta),$$
$$\delta_1\beta = +\,0''.0383 \sin \zeta''' \qquad - 0''.0012 \sin(\zeta''' - 2\tau).$$

The action of Saturn produces the terms

$$\delta\lambda = +\,0''.0002 \sin(\zeta^{iv} - \eta) - 0''.0001 \sin(\zeta^{iv} + \eta),$$
$$\delta_1\beta = +\,0''.0031 \sin \zeta^{iv}.$$

The terms having the same period in the indirect and direct actions of the planets may be united in a single term, and we have

$$\zeta - \eta = \Omega - \Pi + 90^\circ, \qquad \zeta + \eta = 2\,\mathbb{C} - \Omega - \Pi + 90^\circ;$$
$$\zeta' - \eta = \Omega - \Omega'', \qquad \zeta' + \eta = 2\,\mathbb{C} - \Omega - \Omega''.$$

Thus, preserving only the terms whose coefficients exceed 0''.01, the value of $\delta\lambda$ due to both the indirect and direct action of the planets, is

$$\begin{aligned}\delta\lambda &= +\,0''.0305 \sin\Omega - 0''.2838 \cos\Omega \\ &\quad + 0''.0100 \sin(2\,\mathbb{C} - \Omega) - 0''.0697 \cos(2\,\mathbb{C} - \Omega). \\ &= +\,0''.2854 \sin(\Omega + 276^\circ\ 8') + 0''.0704 \sin(2\,\mathbb{C} - \Omega + 278^\circ),\end{aligned}$$

In the case of the latitude we may write the true orbit longitude L of the moon in the place of the mean in the principal term and neglect the remaining terms. Thus the value of $\delta_1\beta$, due to both actions of the planets, is

$$\begin{aligned}\delta_1\beta &= -\,0''.2256 \sin L + 1''.5802 \cos L \\ &= +\,1''.5963 \sin(L + 98^\circ\ 8'.6).\end{aligned}$$

The terms in $\delta\lambda$ and $\delta_1\beta$ which involve $\sin\Omega$ and $\sin L$ coalesce with the principal inequalities which are due to the figure of the earth and have the same arguments. Hansen (*Tables de la Lune*, pp. 8, 15) has, respectively in the perturbed mean anomaly and latitude, the terms $+\,7''.760 \sin(184^\circ\ 42' - \Omega)$ and $+\,8''.764 \sin(L + 169^\circ\ 51')$. The parts of these which depend on $\cos\Omega$ and $\cos L$ are $-\,0''.636 \cos\Omega$ and $+\,1''.544 \cos L$. In the *Darlegung* he gives coefficients somewhat different. As to $\delta_1\beta$, Hansen's value nearly coincides with mine, but his coefficient in $\delta\lambda$ is more than double mine. This discrepancy is probably to be attributed to difference of the system of co-ordinates employed.

The values of these terms which Sir G. B. Airy has determined from obser-

vation, in his first memoir on the correction of the lunar elements (*Mem. Astr Soc.*, Vol. XVII), are

$$\delta\lambda = -0''.97 \cos \Omega, \qquad \delta_1\lambda = +2''.17 \cos L.$$

These he has changed to

$$\delta\lambda = -1''.06 \cos \Omega, \qquad \delta_1\lambda = +1''.93 \cos L,$$

in his second memoir (*Mem. Astr. Soc.*, Vol. XXIX).

SYLVESTER'S TERNARY CUBIC FORM.

By Mr. A. M. SAWIN, Madison, Wis.

Having noticed with interest some of the deductions of Professor Sylvester from what he denominates his Ternary Cubic,* I desire to add a few observations upon this subject and to call attention to some peculiar properties of this cubic which I believe have not been alluded to. The ternary cubic $x^3 - 3xy^2 \pm y^3$ is described by Professor Sylvester† as exhibiting the property, that, if any prime number of the form $18n \pm 1$ or $3(18n \pm 1)$, where n is zero or some integer, can be represented by integer values of x, y in the above expression, then is that number or its nonuple the sum of two cubes. As a matter of fact all such numbers below 537 inclusive (39 in all) were ascertained by Professor Sylvester to be thus representable by integer values of x, y, which he presumed according to the usual canons of induction were sufficient to generalize this remarkable property of the above ternary cubic form to all numbers.‡ Although these actual verifications, are far from establish the generality of the expression for all such numbers; nevertheless, we shall see that this cubic does possess some interesting general properties.

THEOREM I.

If in the equation $x^3 - 3xy^2 \pm y^3 = 18n \pm 1$ or $3(18n \pm 1)$, x and y be any two numbers prime to each other, then will n be zero or some integer.

LEMMA I.

The difference of any two numbers prime to each other is prime to either.

Let x and y represent any two numbers prime to each other; and put x

American Journal of Mathematics, Vol. II. p. 280.

† The same, Vol. II. p. 283.

‡ The same, Vol. III. p. 58.

equal line AB and y equal line CD; then will their difference ED be prime to AB and CD. If ED contains any factor contained an integral number of times in AB, it will also be contained an integral number of times in CE, and would also be contained an integral number of times in CD. Hence AB and CD would have a common factor, which is contrary to the hypothesis; therefore ED is prime to AB and CD. *Q. E. D.*

LEMMA II.

If in the equation $x^3 - 3xy^2 \pm y^3 = 18n \pm 1$ or $3(18n \pm 1)$ there be put for one of the quantities their difference, then will the form of the equation be unchanged and the equation be true.

For x put x, and for y put $x - y$; when

$$x^3 - 3xy^2 + y^3 = 18n \pm 1 \text{ or } 3(18n \pm 1)$$

becomes

$$y^3 - 3yx^2 + x^3 = 18n \pm 1 \text{ or } 3(18n \pm 1).$$

And again put $x - y$ for x and y for y; when

$$x^3 - 3xy^2 - y^3 = 18n \pm 1 \text{ or } 3(18n \pm 1)$$

becomes

$$y^3 - 3yx^2 + x^3 = 18n \pm 1 \text{ or } 3(18n \pm 1).$$

LEMMA III.

If in the equation $x^3 - 3xy^2 \pm y^3 = 18n \pm 1$ or $3(18n \pm 1)$ there be put for one of the quantities their sum, then will the form of the equation be unchanged and the equation be true.

For x put x, and for y put $x + y$; when

$$x^3 - 3xy^2 + y^3 = 18n \pm 1 \text{ or } 3(18n \pm 1)$$

becomes

$$y^3 - 3yx^2 - x^3 = 18n \pm 1 \text{ or } 3(18n \pm 1).$$

And again put $x + y$ for x and y for y; when

$$x^3 - 3xy^2 - y^3 = 18n \pm 1 \text{ or } 3(18n \pm 1)$$

becomes

$$y^3 - 3yx^2 - x^3 = 18n \pm 1 \text{ or } 3(18n \pm 1).$$

LEMMA IV.

By an infinite number of applications of Lemmas II and III all numbers prime to each other are compared in pairs.

Put $x = 1, y = 1$; then by successive applications of Lemma III we shall have

System (1)

$x = 1$	$y = 1$	case (1)
$x = 1$	$y = 2$	" (2)
$x = 1$	$y = 3$	" (3)
$x = 1$	$y = 4$	" (4)
$x = 1$	$y = 5$	" (5)
etc.	etc.	etc.

to infinity,

and for each case the expression $x^3 - 3xy^2 \pm y^3$ will equal $18n \pm 1$ or $3(18n \pm 1)$. Thus 1 is compared with all the numbers prime to it when system (1) is infinite. To pass to system (2) or that system where 2 is compared with all the numbers prime to it, treat case (2) system (1) by Lemma III, thus:—

System (2)

$y = 2$	$x = 1$	case (1)
$y = 2$	$x = 3$	" (2)
$y = 2$	$x = 5$	" (3)
$y = 2$	$x = 7$	" (4)
$y = 2$	$x = 9$	" (5)
etc.	etc.	etc.,

whence 2 is compared with all the numbers prime to it. To obtain system (3), treat case (3) system (1) by composition (Lemma III), thus:—

System (3)

$y = 3$	$x = 1$
$y = 3$	$x = 4$
$y = 3$	$x = 7$
$y = 3$	$x = 10$
$y = 3$	$x = 13$
etc.	etc.

To complete this system, evidently we must interpolate another system. Thus: $y = 3, x = 5$; $y = 3, x = 8$; $y = 3, x = 11$; etc., do not appear in the above system. To reach these cases, treat case (2) system (2) by Lemma III, thus:—

System (3)

$x = 3$	$y = 5$
$x = 3$	$y = 8$
$x = 3$	$y = 11$
$x = 3$	$y = 14$
etc.	etc.,

and system (3) is complete, and 3 is compared with all the numbers prime to it. Similarly treating case (4) system (1) and interpolating from systems (2) and (3), we may obtain system (4) complete. Likewise obtaining system (5) and so on for the other systems, all numbers prime to each other are compared in pairs where the number of systems is infinite. Whence, Lemmas I, II, III, and IV, Theorem I, are true.

We give at random a couple of examples of this theorem. Put $x = 3$, $y = 71$; then

$$3^3 - 3.3.71^2 + 71^3 = 18.17365 - 1.$$

Put $x = 95, y = 4$; then

$$95^3 - 3.95.4^2 + 4^3 = 3\,(18.15794 + 1).$$

THEOREM II.

If in the equation $x^3 - 3xy^2 \pm y^3 = 18n \pm 1$ or $3\,(18n \pm 1)$, where $18n \pm 1$ or $3\,(18n \pm 1)$ represents a prime number, n being zero or some integer and x and y integers, then will x and y be prime to each other.

If they are not prime to each other, assume k equal to their greatest common divisor; so that

$$x = x'k, y = y'k,$$

and substitute and divide by k^3; then

$$x'^3 - 3x'y'^2 \pm y'^3 = \frac{18n \pm 1}{k^3} \text{ or } \frac{3\,(18n \pm 1)}{k^3} = \text{integer}$$
$$= 18n' \pm 1 \text{ or } 3\,(18n' \pm 1).$$

But since $18n \pm 1$ or $3\,(18n \pm 1)$ is a prime,* and $\frac{18n \pm 1}{k^3}$ or $\frac{3\,(18n \pm 1)}{k^3}$ is an integer (Theorem I), evidently $k = 1$; whence x and y are prime to each other, their greatest common divisor being unity, which was to be proved.

Cor. Multiplying the above ternary cubic by k^3 and subsituting x for $x'k$ and y for $y'k$ we have $x^3 - 3xy^2 \pm y^3 = k^3\,(18n' \pm 1)$ or $3k^3\,(18n' \pm 1)$, where x and y have *G. C. D.* equal k. If $k^3 = 18n'' \pm 1$ evidently

$$x^3 - 3xy^2 \pm y^3 = (18n'' \pm 1)\,(18n' \pm 1) \text{ or } 3\,(18n'' \pm 1)\,(18n' \pm 1)$$
$$= 18n \pm 1 \text{ or } 3\,(18n \pm 1).$$

Therefore, Theorem I may receive this enlargement in the case of composites;

*The possibility of the numbers being divisible by 3 as it may be $3\,(18n \pm 1)$ is the obvious exception to this theorem.

viz.: x and y may have a *G. C. D.* which, cubed, equals the canonical expression $18n'' \pm 1$. Evidently k equals the series of numbers $6m \pm 1$, as will be seen from the expansion $k^3 = 18(12m^3 \pm 6m^2 + m) \pm 1$ or $k^3 = 18n'' \pm 1$ where m is zero or some integer.

From Theorem I we may now obtain the general theorem of which Professor Sylvester's expression $x^3 - 3xy^2 \pm y^3$ is a particular case. If the general binary form

$$px^n + qx^{n-1}y + rx^{n-2}y^2 + sx^{n-3}y^3 + tx^{n-4}y^4 + \ldots + wy^n$$

be given and in accordance with Lemma III we put x for x and $x + y$ for y, then evidently

$$\begin{aligned}
px^n &\text{ becomes } px^n,\\
qx^{n-1}y &\text{ becomes } qx^n + qx^{n-1}y,\\
rx^{n-2}y^2 &\text{ becomes } rx^n + 2rx^{n-1}y + rx^{n-2}y^2,\\
&\ldots\ldots\ldots\ldots\ldots\\
vxy^{n-1} &\text{ becomes } vx^n + (n-1)vx^{n-1}y + \frac{(n-1)(n-2)}{2}vx^{n-2}y^2\\
&\qquad + \frac{(n-1)(n-2)(n-3)}{2.3}vx^{n-3}y^3 + \ldots,\\
wy^n &\text{ becomes } wx^n + nwx^{n-1}y + \frac{n(n-1)}{2}wx^{n-2}y^2 + \ldots + wy^n.
\end{aligned}$$

Impose the condition that the variables in the transformed expression shall simply interchange places, no change taking place in the coefficients, and we shall obtain

$$\begin{aligned}
p + q + r + \ldots + w &= w\\
q + 2r + 3s + \ldots + nw &= v\\
r + 3s + \ldots + \frac{n(n-1)}{2}w &= u\\
s + \ldots + \frac{n(n-1)(n-2)}{2.3}w &= t,\\
&\ldots\ldots\ldots\\
w &= p
\end{aligned}$$

From that system of homogeneous equations determine the values of p, q, r, s, etc., and substitute in the binary form we obtain the general expression for the binary form of any degree enjoying like properties with Professor Sylvester's ternary cubic; in fact that expression, $x^3 - 3xy^2 \pm y^3$, is a particular case of the general form. Let it be required, for example, to find this analogue of the fourth degree. Evidently $n = 4$, and

$$\begin{aligned} p+q+r+s+t &= t,\\ q+2r+3s+4t &= s,\\ r+3s+6t &= r,\\ s+4t &= q,\\ t &= p, \end{aligned}$$

whence

$$p=1,\ q=2,\ r=-1,\ s=-2,\ t=1.$$

Substituting these values in the general binary equation, we have

$$x^4+2x^3y-x^2y^2-2xy^3+y^4=24n\pm 1 \text{ or } 4(24n\pm 1).$$

The same general properties, Theorems I and II, may be predicated of this quartic as of the above cubic, inclusive of its enlargement for the composite numbers *G. C. D.* equal $6m\pm 1$. Examples: —

Assume x, y equal any two numbers prime to each other, $x=4, y=31$. Evidently

$$4^4+2.4^3.31-4^2.31^2-2.4.31^3+31^4=24.28085+1,\ n=28085.$$

Assume $x=5, y=15$. *G. C. D.* $=5$, $m=1$.
Evidently

$$625+3750-5625-33750+50625=24.651+1,\ n=651.$$

SOLUTIONS OF EXERCISES.

2

THREE closely connected tanks, T_1, T_2, T_3, contain Q_1 gallons of water, Q_2 gallons of vinegar, Q_3 gallons of brandy, respectively. A flow is set up from T_1 through T_2 to T_3 and back to T_1 at the rate of 1 gallon per second. The liquids are assumed to mix instantaneously, and the lengths of the connecting pipes are neglected. Show how to calculate the amount of water in each tank at the end of t seconds. [*DeVolson Wood.*]

SOLUTION.

Let x_1, x_2, and x_3 be the quantities of water in the three tanks at the time t;

then, since $x_1 \div Q_1$ is the quantity of water contained in a unit of the mixture in the tank T_1, etc., we have, putting for convenience a_1, a_2, and a_3 for the reciprocals of Q_1, Q_2, and Q_3, the differential equations

$$\frac{dx_1}{dt} = a_3x_3 - a_1x_1, \text{ etc.};$$

or, putting D for $\frac{d}{dt}$ symbolically,

$$(D + a_1)\,x_1 - a_3x_3 = 0, \tag{1}$$
$$(D + a_2)\,x_2 - a_1x_1 = 0, \tag{2}$$
$$(D + a_3)\,x_3 - a_2x_2 = 0. \tag{3}$$

Solving for x_1 we find

$$\begin{vmatrix} D + a_1 & . & -a_3 \\ -a_1 & D + a_2 & . \\ . & -a_2 & D + a_3 \end{vmatrix} x_1 = 0$$

or

$$D[D^2 + (a_1 + a_2 + a_3)\,D + a_1a_2 + a_2a_3 + a_3a_1]\,x_1 = 0. \tag{4}$$

The same equation exists for x_2 and x_3, the values differing from that of x_1 only in the constants of integration. The complete integral of (4) may be written in the form

$$a_1x_1 = Ae^{\lambda_2 t} + Be^{\lambda_1 t} + C, \tag{5}$$

where λ_1 and λ_2 are the roots of the quadratic

$$\lambda^2 + (a_1 + a_2 + a_3)\,\lambda + a_1a_2 + a_2a_3 + a_3a_1 = 0; \tag{6}$$

then from (1)

$$a_3x_3 = \frac{\lambda_1 + a_1}{a_1} Ae^{\lambda_1 t} + \frac{\lambda_2 + a_1}{a_1} Be^{\lambda_2 t} + C, \tag{7}$$

whence from (3)

$$a_2x_2 = \frac{\lambda_1 + a_3}{a_3}\frac{\lambda_1 + a_1}{a_1} Ae^{\lambda_1 t} + \frac{\lambda_2 + a_3}{a_3}\frac{\lambda_2 + a_1}{a_1} Be^{\lambda_2 t} + C. \tag{8}$$

To determine the constants we have when $t = 0$, $x_1 = Q_1$, $x_2 = 0$, $x_3 = 0$; hence (5), (7), and (8) give

$$1 = A + B + C,$$
$$0 = \frac{\lambda_1 + a_3}{a_1} A + \frac{\lambda_2 + a_1}{a_1} B + C,$$
$$0 = \frac{\lambda_1 + a_3}{a_3}\frac{\lambda_1 + a_1}{a_1} A + \frac{\lambda_2 + a_3}{a_3}\frac{\lambda_2 + a_1}{a_1} B + C;$$

from which we obtain

$$-1 = \frac{\lambda_1}{a_1} A + \frac{\lambda_2}{a_1} B,$$

and

$$0 = \frac{\lambda_1}{a_3} \frac{\lambda_1 + a_1}{a_1} A + \frac{\lambda_2}{a_3} \frac{\lambda_2 + a_1}{a_1} B.$$

Whence

$$A = \frac{a_1 (\lambda_2 + a_1)}{\lambda_1 (\lambda_1 - \lambda_2)},$$
$$B = \frac{a_1 (\lambda_1 + a_1)}{\lambda_2 (\lambda_2 - \lambda_1)},$$
$$C = \frac{\lambda_1 \lambda_2 + a_1 (\lambda_2 + \lambda_1 + a_1)}{\lambda_1 \lambda_2},$$

which by (6) reduces to

$$C = \frac{a_2 a_3}{a_1 a_2 + a_2 a_3 + a_3 a_1},$$

Hence

$$x_1 = \frac{\lambda_2 + a_1}{\lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{\lambda_1 + a_1}{\lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} + \frac{C}{a_1},$$
$$x_3 = \frac{(\lambda_1 + a_1)(\lambda_2 + a_1)}{a_3 \lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{(\lambda_2 + a_1)(\lambda_1 + a_1)}{a_3 \lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} + \frac{C}{a_3},$$
$$x_2 = \frac{(\lambda_1 + a_3)(\lambda_1 + a_1)(\lambda_2 + a_1)}{a_2 a_3 \lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{(\lambda_2 + a_3)(\lambda_2 + a_1)(\lambda_1 + a_1)}{a_2 a_3 \lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} + \frac{C}{a_2},$$

where λ_1 and λ_2 are

$$-\frac{1}{2}(a_1 + a_2 + a_3) \pm \frac{1}{2} \sqrt{(a_1^2 + a_2^2 + a_3^2 - 2a_1 a_2 - 2a_2 a_3 - 2a_3 a_1)},$$

and C is as above. Adding these values, it will be found that

$$x_1 + x_2 + x_3 = \frac{1}{a_1} = Q_1,$$

as evidently should be the case.

[*Wm. Woolsey Johnson.*]

3

FIND the radius of a sphere which, let fall into a given conical wine-glass full of water, will displace the maximum or minimum quantity of water.

[*Benjamin Alvord.*]

SOLUTION.

Let x be the radius of the sphere, y the distance of its centre from the surface of the water. The volume displaced will be

$$V = \frac{\pi}{3}(x-y)^2(2x+y).$$

But if a is the radius of the top, b the depth, and c the slant height of the cone

$$y = \frac{cx - ab}{a};$$

whence

$$V = \frac{\pi}{3a^3}(ax - cx + ab)^2(2ax + cx - ab).$$

This will have its maximum or minimum value when

$$f'(x) = 2(ax - cx + ab)(2ax + cx - ab)(a - c) + (ax - cx + ab)^2(2a + c) = 0.$$

Two cases then arise.

1. The first is given by putting

$$ax - cx + ab = 0.$$

The corresponding value of V is a minimum; but the circle of contact of the sphere with the cone is above the top of the glass; so that this solution must be rejected.

2. The second is given by putting

$$2(2ax + cx - ab)(a - c) + (ax - cx + ab)(2a + c) = 0,$$

whence

$$x = \frac{abc}{(c + 2a)(c - a)}.$$

The corresponding value of V is a maximum, as is shown by substitution in $f''(x)$. [*W. F. C. Hasson.*]

4

On one of the bounding radii of a given quadrant, radius r, a semicircle is drawn, radius $\frac{1}{2}r$, the semicircle being within the quadrant. Find the average area of the circle touching the other bounding radius of the quadrant and the arc of the semicircle. [*Artemas Martin.*]

SOLUTION BY THE PROPOSER.

Let x and y be the co-ordinates of a point on the locus of the centres of

the tangent circles, the axis of x being the diameter of the semicircle, and that of y being the other bounding radius of the quadrant; then we have

$$\left(\tfrac{1}{2}r - x\right)^2 + y^2 = \left(\tfrac{1}{2}r + x\right)^2;$$

whence

$$y^2 = 2rx. \qquad (1)$$

Therefore the centres of all the circles are located on the arc of a parabola whose vertex is at the origin of co-ordinates. The *largest* circle is that which also touches the arc of the quadrant. We then have the additional equation

$$y^2 + x^2 = (r - x)^2,$$

whence

$$y^2 = r^2 - 2rx. \qquad (2)$$

From this and (1) we get

$$y = \tfrac{1}{2}r\sqrt{2},$$

which is the superior limit of y.

Let s = length of the parabolic arc, estimated from the origin of co-ordinates, and A = the average area required; then, the area of the circle being $\pi x^2 = \dfrac{\pi y^4}{4r^2}$,

$$A = \frac{\int_{y=0}^{y=\frac{1}{2}r\sqrt{2}} \frac{\pi y^4}{4r^2}\, ds}{\int_{y=0}^{y=\frac{1}{2}r\sqrt{2}} ds}$$

But by (1)

$$ds = \sqrt{(dx^2 + dy^2)} = \frac{1}{r}\sqrt{(y^2 + r^2)}\, dy.$$

$$\therefore A = \frac{\frac{\pi}{4r^3}\int_0^{\frac{1}{2}r\sqrt{2}} y^4\sqrt{(r^2 + y^2)}\, dy}{\frac{1}{r}\int_0^{\frac{1}{2}r\sqrt{2}} \sqrt{(r^2 + y^2)}\, dy}$$

$$= \frac{1}{32}\pi r^2 \left\{ 1 - \frac{1}{1\,3 + 2\log\left(\frac{1}{2}1\,2 + \frac{1}{2}1\,6\right)} \right\}.$$

5

THREE lines are drawn from a point within a square to three corners of the square. The lengths of these lines are given. Find a side of the square.

[*J. M. Greenwood.*]

SOLUTION I.

Let a, b, c, be the lengths of the lines. On the sides of the right angle A lay off AP, AQ, each equal to a, and join PQ. On PQ as base with bc as sides construct the triangle PQR, and join AR. Then AR is the side of the required square. [*Alexander Evans.*]

SOLUTION II.

Refer the points to the diagonals of the square as axes, and let x, y be its co-ordinates. Let l be the length of the half-diagonal. Then

$$x^2 + (l - y)^2 = a^2,$$
$$x^2 + (l + y)^2 = b^2,$$
$$y^2 + (l \pm x)^2 = c^2;$$

whence for l the biquadratic

$$8l^4 - 4(a^2 + b^2)l^2 + a^4 + b^4 + 2c^4 - 2a^2c^2 - 2b^2c^2 = 0.$$

The side of the square is $l\sqrt{2}$. [*E. W. Morley.*]

SOLUTION III.

Let AB, BC be sides of the square, and M the point. Put $AB = s$; $MA, MB, MC = a, b, c$. Then

$$\cos ABM = \frac{s^2 + b^2 - a^2}{2bs},$$
$$\cos MBC = \frac{s^2 + b^2 - c^2}{2bs};$$

and, as these angles are complementary,

$$(s^2 + b^2 - a^2)^2 + (s^2 + b^2 - c^2)^2 = 4b^2s^2,$$

a quadratic in s^2 from which the length of the side is readily computed.

[*L. G. Barbour.*]

8

IF an ellipse and a rectangular hyperbola have the same centre, and the hyperbola passes through the focus of the ellipse, then at the point of intersec-

tion of the curves the ellipse makes equal angles with the hyperbola and the central radius. [*H. A. Newton.*]

SOLUTION I.

The rectangular hyperbola may always be regarded as generated by two equal (projective) pencils of opposite directions of rotation. In the present case the two foci F_1, F_2 of the ellipse, being points of the hyperbola, may be assumed as the centres of such pencils. To completely determine the position of the hyperbola, it only remains to assume an arbitrary ray F_2A of the pencil F_2 as corresponding to the ray F_1F_2 of the pencil F_1. The asymptotes will then be parallel to the bisectors of the angles formed by F_2F_1 and F_2A.

If P be one of the points of intersection of the two curves, and Q the point where the bisector of $\angle AF_2F_1$ meets F_1P, we evidently have

$$\angle PF_1F_2 = \angle PF_2A,$$

and hence

$$\angle PQF_2 = \angle PF_2Q.$$

A line through P parallel to F_2Q (*i. e.* parallel to the asymptote) will therefore make equal angles with PF_1 and PF_2, that is, will be tangent to the ellipse in P. Thus we see that the tangent and normal of the ellipse in P are parallel to the asymptotes of the hyperbola. This, in connection with the property of the hyperbola that the segment of the tangent intercepted by the asymptotes is bisected in the point of contact, proves the proposition. [*Alexander Ziwet.*]

SOLUTION II.

Let the equations of the ellipse and rectangular hyperbola be

$$x^2 + \frac{y^2}{m} = c, \tag{1}$$

$$x^2 - y^2 - 2pxy = c(1 - m), \tag{2}$$

where (2), being satisfied by $x = \sqrt{c(1-m)}$, $y = 0$, passes through the focus of of the ellipse. Combining (1) and (2), we get for their point of intersection

$$x^2 - m(c - x^2) - 2px\sqrt{m(c - x^2)} = c(1 - m). \tag{3}$$

Now, if $x'y'$ denote this point, we have for the equation to the central radius

$$y = \frac{y'}{x'}x; \tag{4}$$

for the equation to the tangent to the ellipse,

$$y = -\frac{mx'}{y'}x + mc; \tag{5}$$

for the equation to the tangent to the hyperbola,

$$xx' - p(x'y + y'x) - yy' = c(1 - m)$$

or

$$y = \frac{x' - py'}{y' + px'}\,x - \frac{c(1-m)}{y' + px'}. \qquad (6)$$

Let α equal the angle between the central radius and the tangent to the ellipse, and β equal the angle between the tangents to the ellipse and the hyperbola; then

$$\tan \beta = \frac{x'y' - py'^2 + mx'y' + mpx'^2}{y'^2 + px'y' - mx'^2 + mpx'y'},$$

$$\tan \alpha = \frac{mx'^2 + y'^2}{(m-1)x'y'}.$$

That $\tan \alpha$ may equal $\tan \beta$, we must have, discarding the primes,

$$(mx^2 + y^2)(y^2 + pxy - mx^2 + mpxy) = (mxy - xy)(xy - py^2 + mxy + mpx^2),$$

which reduces to

$$y^4 + 2mpx^2y + 2mpxy^3 - m^2x^4 - m^2x^2y^2 + x^2y^2 = 0, \qquad (7)$$

or, substituting for y its value from (1),

$$m^2(c - x^2)^2 + 2mpx^3\sqrt{m(c-x^2)} + 2m^2px(c-x^2)\sqrt{m(c-x^2)},$$
$$- m^2x^4 - m^3x^2(c-x^2) + mx^2(c-x^2) = 0. \qquad (8)$$

From (3),

$$2px\sqrt{m(c-x^2)} = (1+m)x^2 - c.$$

Substituting in (8), we get

$$m^2(c-x^2)^2 + mx^2(x^2 + mx^2 - c) + m^2(c-x^2)(x^2 + mx^2 - c) - m^2x^4 - m^3cx^2$$
$$+ m^3x^4 + mcx^2 - mx^4 = 0,$$

an identity. Therefore $\tan \alpha = \tan \beta$, and $\alpha = \beta$. Q. E. D.

[*S. M. Barton.*]

EXERCISES.

13

A THIN inextensible cord in which the density of the material increases in geometric progression, as the distance from one end increases in arithmetic pro-

gression is laid directly across a rough horizontal cylinder the circumference of a normal section of which is equal to twice the length of the cord; to determine the co-efficient of friction, supposing the cord to be only just supported when its extremities are in the horizontal plane containing the axis of the cylinder. [*William Hoover.*]

14

A FAISCEAU of parabolas can be drawn having the pole of a cardioid as the common focus, all passing through one point and all cutting the cardioid at right angles. [*H. A. Newton.*]

15

IT IS required to inscribe a rectangle of breadth nD within another rectangle whose diagonal is B. Show that the excess E of the declivity M of the diagonal of the greater rectangle over that of the length of the less may be calculated when n is small, from the formula

$$\sin E = n \cos 2M [1 + 2n \sin 2M + n^2 (3 - \cos 2M) + \ldots].$$

[*W. M. Thornton.*]

16

ELIMINATE x from the equations

$$x^6 + ax^2 + bx + c = 0,$$
$$x^3 + dx^2 + ex + f = y,$$

and arrange the result according to powers of y. [*Mansfield Merriman.*]

17

IN HIS work, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, p. 65, Grassmann says: "Lagrange führt in seiner *Méc. Anal.*, p. 14 der neuen Ausgabe, einen Satz von Varignon an, dessen er sich zur Verknüpfung der verschiedenen Principien der Statik bedient. * * * * Dieser Satz ist, wie sich sogleich zeigen wird, unrichtig."

In what way is this theorem incorrect as used by Todhunter and others?

[*Asaph Hall.*]

18

A FIXED point of light shines upon a plane which rotates about a fixed axis with a constant angular velocity. A normal to this plane moves at a constant rate (measured in the plane) in the direction of the shadow which the normal casts upon the plane. Given the position of the normal at a time when the light is in the plane; when the light is again in the plane, how far will the normal have moved during the intervening time? [*F. P. Leavenworth.*]

NOTES.

1

PROF. SMITH remarks with reference to his paper on Rectilinear Asymptotes that the last line of article 4 should read "and β_∞ from $\beta = -\frac{f_1}{\frac{df}{d\mu}}$;" and that in editorial note on this article β_∞ must be determined, not as in the passage cited, but from the equation

$$\frac{d^2 f}{d\mu^2}\beta^2 + 2\frac{df_1}{d\mu}\beta + 2f_2 = 0.$$

2

PROF. G. S. ELY, of Buchtel College, Ohio, sends the following properties of the periodic continued fractions equivalent to the square roots of integers. Such fractions are written with the denominators of the repetend in square brackets annexed to the integral part. Thus

$$\sqrt{7} = 2[1, 1, 1, 4] = 2 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+\ldots}.$$

1. If $2n$ contains p,

$$\sqrt{(n^2 + p)} = n\left[\frac{2n}{p}, 2n\right],$$

$$\sqrt{(n^2 - p)} = (n-1)\left[1, \frac{2n}{p} - 2, 1, 2n-2\right].$$

2. If $n - 1$ contains 2,

$$\sqrt{(n^2 + 4)} = n\left[\frac{1}{2}(n-1), 1, 1, \frac{1}{2}(n-1), 2n\right],$$

$$\sqrt{(n^2 - 4)} = (n-1)\left[1, \frac{1}{2}(n-3), 2, \frac{1}{2}(n-3), 1, 2n-2\right].$$

3. If $n - 1$ contains 3,

$$\sqrt{(n^2 + n + 1)} = n\left[1, 1, \frac{2n-2}{3}, 1, 1, 2n\right],$$

$$\sqrt{\left(n^2 + \frac{2n+1}{3}\right)} = n[2, 1, n-1, 1, 2, 2n],$$

$$\sqrt{\left(n^2 + \frac{4n+2}{3}\right)} = n[2, 1, n, 1, 2, 2n].$$

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